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Construction of the viability kernel for a generalized dynamical system[☆]

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Abstract

The problem of the approximate construction of the viability kernel for a generalized dynamical system, the evolution of which is specified directly by an attainability set, is investigated under phase constraints. A backward grid method, based on the substitution of the phase space by pixels and a consideration of "inverse" attainability sets, is proposed. The convergence of the method is proved.

Generalized dynamical systems – the result of an axiomatic approach to control systems – have been investigated intensively.[1–6] Investigations in the theory of controllable systems can be extended to generalized dynamical systems.[7–11,a]

Below, in a continuation of earlier publications (Refs. 12,13), results obtained previously in Ref. 14 are extended to generalized dynamical systems.

1. Formulation of the problem

Consider a generalized dynamical system, the behaviour of which is specified by means of the multivalued mapping

$$F(\cdot) : I \times I \times R^m \to 2^{R^m}, \tag{1.1}$$

where $I = [t_0, \theta]$ is a finite time interval. For specified $(t_*, x_*) \in I \times R^m$ and $t^* \in [t_*, \theta]$ the symbol $F(t^*; t_*, x_*)$ denotes the attainability set of the generalized dynamical system from the initial position (t_*, x_*) at the instant of time t^*. We will assume that the multivalued mapping (1.1) satisfies the following conditions 1°–6°.

1°. The attainability set $F(t^*; t_*, x_*)$ is defined for all $(t_*, x_*) \in I \times R^m$, $t^* \in [t_*, \theta]$ and is a non-empty compactum in the space R^m.

2°. A constant $M > 0$ exists such that for all $(t_*, x_*) \in I \times R^m$, $\delta \in [0, \theta - t_*]$ the following inequality holds

$$\alpha(\{x_*\}, F(t_* + \delta; t_*, x_*)) \le M\delta.$$

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[a] See also: Filippova TF. Problems on viability for differential inclusions: Doctorate Disseration, 01. 01. 02. Ekaterinburg, 1992.

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The Hausdorf distance between the sets $A \subset R^m$ and $C \subset R^m$ is defined as

$$\alpha(A, C) = \max\{\sup_{x \in A} \operatorname{dist}(x, C), \sup_{y \in C} \operatorname{dist}(y, A)\}; \quad \operatorname{dist}(x, C) = \inf\{\|x - y\| : y \in C\}.$$

Here and below $||\cdot||$ denotes the Euclidean norm.

3°. The following equality holds for any $(t_*, x_*) \in I \times R^m$, t and t^* $(t_* \le t < t^* \le \theta)$

$$F(t^*; t_*, x_*) = \bigcup\{F(t^*; t, x) : x \in F(t; t_*, x_*)\}.$$

4°. For specified $(t^*, x^*) \in I \times R^m$ and $t_* \in [t_0, t^*]$ a point $x_* \in R^m$ exists such that $x^* \in F(t^*; t_*, x_*)$.
5°. A function $\omega^*(\Delta)$ exists, which tends monotonically to zero when $\Delta \downarrow 0$, and such that

$$\alpha(F(t_1 + \delta; t_1, x_1) - x_1, F(t_2 + \delta; t_2, x_2) - x_2) \le \delta\omega^*(|t_1 - t_2| + \|x_1 - x_2\|)$$

$$\forall (t_1, x_1), (t_2, x_2) \in I \times R^m, \quad \forall \delta \in [0, \theta - \max(t_1, t_2)].$$

6°. An $L > 0$ exists such that

$$\alpha(F(t_* + \delta; t_*, x_1) - x_1, F(t_* + \delta; t_*, x_2) - x_2) \le \delta L\|x_1 - x_2\|$$

$$\forall (t_*, x_1), (t_*, x_2) \in I \times R^m, \quad \forall \delta \in [0, \theta - t_*].$$

We will state an assertion characterizing certain properties of generalized dynamical system (1.1).

Assertion 1.1. For all $(t_*, x_*) \in I \times R^m$:

1) the equality $F(t_*; t_*, x_*) = \{x_*\}$ holds;
2) the multivalued mapping $\delta \to F(t_* + \delta; t_*, x_*)$ in the interval $[0, \theta - t_*]$ is continuous in the Hausdorf metric and satisfies the inequality

$$\alpha(F(t_* + \delta_1; t_*, x_*), F(t_* + \delta_2; t_*, x_*)) \le M|\delta_1 - \delta_2|, \quad \forall \delta_1, \delta_2 \in [0, \theta - t_*].$$

The attainability sets $F(t^*; t_*, x_*)$, for example, of the differential inclusion

$$\dot{x} \in \operatorname{co}\{f(t, x, u) : u \in P\}, \quad t \in I, \quad x \in R^m, \tag{1.2}$$

satisfy properties 1°–6°, where P is a compactum of the space of controls R^p, while the vector function $f(t, x, u)$ satisfies the following two conditions:

1) the function $f(t, x, u)$ is continuous in the set of variables and a constant $L_1 \ge 0$ exists such that

$$\|f(t, x_1, u) - f(t, x_2, u)\| \le L_1\|x_1 - x_2\|, \quad \forall (t, x_1, u), (t, x_2, u) \in I \times R^m \times P.$$

2) a constant $M_1 \ge 0$ exists such that

$$\|f(t, x, u)\| \le M_1, \quad \forall (t, x, u) \in I \times R^m \times P.$$

Definition 1.1 *((Refs. 1,2)).* Any function $x(\cdot)$: $[t_*, \theta] \to R^m$, $x(t_*) = x_*$ which satisfies the inclusion

$$x(t_2) \in F(t_2; t_1, x(t_1)), \quad \forall t_1, t_2 \in I, \quad t_1 < t_2$$

will be called a trajectory of the generalized dynamical system (1.1), emerging from the initial position $(t_*, x_*) \in I \times R^m$. The set of such trajectories will be denoted by the symbol $X(t_*, x_*)$.

We will also put

$$X(t; t_*, x_*) = \{x(t) \in R^m : x(\cdot) \in X(t_*, x_*)\}.$$

It is well known[1,2] that any trajectory $x(\cdot) \in X(t_*, x_*)$ is continuous in the interval $[t_*, \theta]$, the set $X(t_*, x_*)$ is closed and the following equalities are satisfied

$$X(t; t_*, x_*) = F(t; t_*, x_*), \quad \forall (t_*, x_*) \in I \times R^m, \quad \forall t \in [t_*, \theta].$$

Suppose that, in addition to the generalized dynamical system (1.1), we are given the closed set $\Phi \subset I \times R^m$, which has non-empty sections $\Phi(t) = \{x \in R^m\colon (t, x) \in \Phi\}$ $(t \in I)$. Suppose $\Phi(\theta)$ is a compactum in R^m.

We will say that the trajectory $x(\cdot) \in X(t_*, x_*)$ is viable in the set Φ if the inclusion $(t, x(t)) \in \Phi$ is satisfied for all $t \in [t_*, \theta]$.

Definition 1.2. We will call the set of all points $(t_*, x_*) \in I \times R^m$ for which the trajectory $x(\cdot) \in X(t_*, x_*)$, viable in Φ, exists, the viability kernel Ω of the generalized dynamical system (1.1) in the set Φ.

Obviously $\Omega \subset \Phi$.

We will investigate the problem of the approximate construction of the kernel Ω.

2. Time discretization

We will replace the time interval I by a finite set of instants. We will specify the sequence $\{\Gamma_n\}$ of subdivisions

$$\Gamma_n = (t_0 = t^0, t^1, \ldots, t^n = \theta)$$

of the section I with diameter $\Delta_n = t^{i+1} - t^i$ $(i = 0, 1, \ldots n-1)$, satisfying the relation $\Delta_n = (\theta - t_0)/n$; here, for each n, there are its own instants t^i of the subdivision Γ_n.

We will assume

$$F^{-1}(t_*; t^*, x^*) = \{x_* \in R^m : x^* \in F(t^*; t_*, x_*)\}$$

$$\tilde{F}^{-1}(t_*; t^*, x^*) = 2x^* - F(t^*; t_*, x^*), \quad \omega(\Delta) = \Delta\omega^*((1+M)\Delta)$$

$$\tilde{F}^{-1}(t_*; t^*, X^*) = \bigcup\{\tilde{F}^{-1}(t_*; t^*, x) : x \in X^*\}.$$

Here $(t^*, x^*) \in I \times R^m$, $t_* \in [t_0, t^*]$, $\Delta > 0$, $X^* \subset R^m$.

The following two assertions hold.

Assertion 2.1. The following inclusion holds

$$F^{-1}(t_*; t^*, x^*) \subset \tilde{F}^{-1}(t_*; t^*, x^*)_{\omega(t^* - t_*)}, \quad \forall (t^*, x^*) \in I \times R^m, \quad \forall t_* \in [t_0, t^*].$$

Here and below F_ε when $\varepsilon \geq 0$ is the closure of the ε-neighbourhood of the set $F \subset R^m$.

The inverse inclusion $\tilde{F}^{-1}(t_*; t^*, x^*) \subset F^{-1}(t_*, t^*, x^*)_\delta$, generally speaking, does not hold for all $\delta \to 0$.

Assertion 2.2. The following inequality holds

$$\alpha(\tilde{F}^{-1}(t^* - \delta; t^*, x_1) - x_1, \tilde{F}^{-1}(t^* - \delta; t^*, x_2) - x_2) \leq \delta L \|x_1 - x_2\|$$

$$\forall (t^*, x_1), (t^*, x_2) \in I \times R^m, \quad \forall \delta \in [0, t^* - t_0].$$

For each subdivision Γ_n there will be a corresponding sequence $\{\tilde{\varepsilon}^i\}$ of numbers, specified recurrently

$$\tilde{\varepsilon}^n = 0; \quad \tilde{\varepsilon}^i = \omega(\Delta_n) + (1 + L\Delta_n)\tilde{\varepsilon}^{i+1}, \quad i = n-1, n-2, \ldots, 0.$$

We will denote the greatest of the numbers of this sequence by the symbol $\tilde{\varepsilon}_n$.

Lemma 2.1. *The following limit relation holds*

$$\lim \tilde{\varepsilon}_n = 0.$$

In fact, it can be shown by induction that the following estimate holds

$$\tilde{\varepsilon}^i \le e^{(n-i)L\Delta_n}(n-i)\omega(\Delta_n), \quad i = 0, 1, \ldots, n,$$

from which we have the inequality

$$\tilde{\varepsilon}_n = \tilde{\varepsilon}^0 \le e^{L(\theta - t_0)}(\theta - t_0)\omega^*((1+M)\Delta_n),$$

which, by virtue of the limit relations

$$\lim \Delta_n = 0, \quad \lim_{\Delta \to 0} \omega^*(\Delta) = 0$$

proves the correctness of this lemma.

For each subdivision Γ_n there will be a corresponding sequence of set $\tilde{\Omega}_n(t^i) \subset R^m (t^i \in \Gamma_n)$, specified by recurrence relations, beginning from the final instant $t^n = \theta$.

Definition 2.1. We will assume that

$$\tilde{\Omega}_n(\theta) = \Phi(\theta)_{\tilde{\varepsilon}^n} = \Phi(\theta)$$

$$\tilde{\Omega}_n(t^i) = \Phi(t^i)_{\tilde{\varepsilon}^i} \cap \tilde{F}^{-1}(t^i; t^{i+1}, \tilde{\Omega}_n(t^{i+1})), \quad i = n-1, n-2, \ldots, 0.$$

We will define the limit of the sequence $\{\tilde{\Omega}_n(t^i)\}$, when the diameter of the subdivision Γ_n approaches zero.

Definition 2.2. We will assume that $\tilde{\Omega}^0$ is a set of all points $(t_*, x_*) \in I \times R^m$, for which we obtain the sequence

$$\{(\eta_n, x_n) : \eta_n = t_n(t_*), x_n \in \tilde{\Omega}_n(\eta_n)\}$$

such that $(t_*, x_*) = \lim(\eta_n, x_n)$.

Here and below $t_n(t_*) = \min(t^i \in \Gamma_n : t_* \le t^i)$; the limit is taken as $n \to \infty$, unless otherwise indicated.
The inclusion $\tilde{\Omega}^0 \subset \Phi$ follows from Definition 2.2.
The set $\tilde{\Omega}^0$ is non-empty, since the equality $\tilde{\Omega}_n(\theta) = \Phi(\theta)$ holds and, consequently, the section $\tilde{\Omega}^0(\theta) = \{x \in R^m : (\theta, x) \in \tilde{\Omega}^0\}$ of the set $\tilde{\Omega}^0$ is non-empty.

Theorem 2.1. *The set $\tilde{\Omega}^0$ is the viability kernel of the generalized dynamical system (1.1) in the set Φ.*

Proof. We will first prove the inclusion $\tilde{\Omega}^0 \subset \Omega$.
We fix an arbitrary point $(t_*, x_*) \in \tilde{\Omega}^0$ when $t_* < \theta$. We obtain the sequence $\{(\eta_n, x_n) : \eta_n = t_n(t_*), x_n \in \tilde{\Omega}_n(\eta_n)\}$ such that

$$(t_*, x_*) = \lim(\eta_n, x_n).$$

Consider the arbitrary number n. We will show that a trajectory

$$x_n(\cdot) \in X(\eta_n, x_n) \tag{2.1}$$

exists which satisfies the inclusions

$$x_n(t^i) \in \tilde{\Omega}_n(t^i)_{\tilde{\varepsilon}^{n-i}}, \quad t^i \in \Gamma_n, \quad t^i \ge \eta_n \tag{2.2}$$

The following inclusion holds (see Definition 2.1)

$$x_n \in \tilde{F}^{-1}(\eta_n; t^{j+1}, \tilde{x}^{j+1}),$$

where $t^{i+1} = \eta_n + \Delta_n \in \Gamma_n$, $\tilde{x}^{j+1} \in \tilde{\Omega}_n(t^{i+1})$. It follows from this inclusion (see the definition of the set $\tilde{F}^{-1}(\cdot)$) that

$$\tilde{x}^{j+1} - x_n \in F(t^{j+1}; \eta_n, \tilde{x}^{j+1}) - \tilde{x}^{j+1}. \tag{2.3}$$

From the inequality (see property 5°)

$$\alpha(F(t^{j+1}; \eta_n, x_n) - x_n, F(t^{j+1}; \eta_n, \tilde{x}^{j+1}) - \tilde{x}^{j+1}) \le \omega(\Delta_n)$$

and relation (2.3) it follows that the following points exist

$$x^{j+1} \in F(t^{j+1}; \eta_n, x_n),$$

which satisfy the inequality

$$\|(x^{j+1} - x_n) - (\tilde{x}^{j+1} - x_n)\| \le \omega(\Delta_n),$$

and so also the inequality

$$\|x^{j+1} - \tilde{x}^{j+1}\| \le \omega(\Delta_n).$$

Hence, we have obtained the point $x^{j+1} \in F(t^{j+1}; \eta_n, x_n)$ which satisfies the inclusion

$$x^{j+1} \in \tilde{\Omega}_n(t^{j+1})_{\tilde{\varepsilon}^{n-(j+1)}}.$$

Replacing η_n by t^{j+1}, x_n by $\tilde{x}^{j+1}$ and repeating the previous constructions, we obtain the point

$$\bar{x}^{j+1} \in F(t^{j+2}; t^{j+1}, \tilde{x}^{j+1}),$$

which satisfies the inequality

$$\|\bar{x}^{j+2} - \tilde{x}^{j+2}\| \le \omega(\Delta_n), \tag{2.4}$$

where $\tilde{x}^{j+2} \in \tilde{\Omega}_n(t^{j+2})$.

It follows from the relation (see property 6°)

$$\alpha(F(t^{j+2}; t^{j+1}, x^{j+1}) - x^{j+1}, F(t^{j+2}; t^{j+1}, \tilde{x}^{j+1}) - \tilde{x}^{j+1}) \le \Delta_n L\tilde{\varepsilon}^{n-(j+1)}$$

that the following point exists

$$x^{j+2} \in F(t^{j+2}; t^{j+1}, x^{j+1}),$$

satisfying the inequality

$$\|(x^{j+2} - x^{j+1}) - (\bar{x}^{j+2} - \tilde{x}^{j+1})\| \le \Delta_n L\tilde{\varepsilon}^{n-(j+1)},$$

and so also the inequality

$$\|x^{j+2} - \bar{x}^{j+2}\| \le (1 + L\Delta_n)\tilde{\varepsilon}^{n-(j+1)}.$$

Hence, bearing inequality (2.4) in mind, we have

$$\|x^{j+2} - \tilde{x}^{j+2}\| \le \tilde{\varepsilon}^{n-(j+2)}.$$

Hence, we obtain the point $x^{j+2} \in F(t^{j+2}; t^{j+1}, x^{j+1})$, which satisfies the inclusion

$$x^{j+2} \in \tilde{\Omega}_n(t^{j+2})_{\tilde{\varepsilon}^{n-(j+2)}}.$$

Continuing this processing up to the instant $t^n = \theta$, we obtain the remaining points $x^i \in F(t^i; t^{i-1}, x^{i-1})$, which satisfy the inclusions

$$x^i \in \tilde{\Omega}_n(t^i)_{\tilde{\varepsilon}^{n-i}}, \quad i = j+3, j+4, \dots, n.$$

We have thereby proved the existence of the required trajectory (2.1), which satisfies inclusions (2.2).

We will now introduce a function which is a continuous extension of the trajectory obtained into the section $[t_*, \theta]$. Suppose

$$y_n(t) = \begin{cases} x_n(\eta_n), & t_* \le t \le \eta_n \\ x_n(t), & \eta_n < t \le \theta. \end{cases}$$

For $n = 1, 2, \dots$, from the uniformly bounded and equicontinuous sequence $\{y_n(t)\}$ we can separate out a uniformly converging subsequence. Without loss of generality, we will assume that the sequence $\{y_n(t)\}$ itself converges in $[t_*, \theta]$ uniformly to a certain function $x(t)$.

It is easy to show that the function $x(\cdot)$ is a trajectory of the generalized dynamical system (1.1): $x(\cdot) \in X(t_*, x_*)$.

We will show that the trajectory $x(\cdot)$ does not leave the phase constraints:

$$x(t) \in \Phi(t), \quad t \in [t_*, \theta]. \tag{2.5}$$

We fix an arbitrary instant $t \in [t_*, \theta]$ and put $\tau_n = t_n(t)$ (everywhere henceforth $n = 1, 2, \dots$). By virtue of the inclusions (see (2.2))

$$x_n(\tau_n) \in \tilde{\Omega}_n(\tau_n)_{\tilde{\varepsilon}_n}, \quad \tilde{\Omega}_n(\tau_n) \in \Phi(\tau_n)_{\tilde{\varepsilon}_n}$$

we obtain the points $\chi_n \in \Phi(\tau_n)$, which satisfy the inequalities

$$\|x_n(\tau_n) - \chi_n\| \le 2\tilde{\varepsilon}_n.$$

Consequently, together with the limit relations

$$x(t) = \lim y_n(t) = \lim y_n(\tau_n) = \lim x_n(\tau_n) \tag{2.6}$$

the following limit relation (see Lemma 2.1) holds

$$\lim x_n(\tau_n) = \lim \chi_n. \tag{2.7}$$

The limit relation

$$\lim \chi_n \in \Phi(t). \tag{2.8}$$

then follows from the convergence $\lim \tau_n = t$, the boundedness of the sequence $\{x_n\}$ and the closedness of the set Φ.

The inclusion (2.5) is proved by virtue of limit relations (2.6)–(2.8).

Hence, for any point $(t_*, x_*) \in \tilde{\Omega}^0$ when $t_* < \theta$, we obtain the trajectory $x(\cdot) \in X(t_*, x_*)$, which is viable in Φ.

It is also obvious that any point $(t_*, x_*) \in \tilde{\Omega}^0$ satisfies the inclusion $(t_*, x_*) \in \Phi$ when $t_* = \theta$.

The inclusion $\tilde{\Omega}^0 \subset \Omega$ is proved.

We will prove the inverse inclusion $\Omega \subset \tilde{\Omega}^0$.

Consider the subdivision Γ_n of the time interval I and all the sections $\Omega(t^i)$ $(t^i \in \Gamma_n)$ of the set Ω, which are non-empty. We will put $T_n = \{t^i \in \Gamma_n : \Omega(t^i) \ne \emptyset\}$; obviously the set T_n possesses the property that if $t^i \in T_n$, then $t^{i+1} \in T_n$.

We will show the correctness of the inclusion

$$\Omega(t^i) \subset \tilde{\Omega}_n(t^i)_{\tilde{\varepsilon}^i}, \quad t^i \in T_n. \tag{2.9}$$

The proof is carried out by induction, beginning from the final instant $t^n = \theta$.

At the instant $t^n = \theta$ the inclusion (2.9) is satisfied by virtue of the equalities $\Omega(\theta) = \Phi(\theta) = \tilde{\Omega}_n(\theta)_{\tilde{\varepsilon}^n}$.

We will assume that at the instant $t^{i+1} \in T_n$ the inclusion $\Omega(t^{i+1}) \subset \tilde{\Omega}_n(t^{i+1})_{\tilde{\varepsilon}^{i+1}}$ is satisfied. We take an arbitrary point $x^i \in \Omega(t^i)$; the following point corresponds to it

$$x^{i+1} \in \Omega(t^{i+1}), \quad x^{i+1} \in F(t^{i+1}; t^i, x^i).$$

Taking the point $\tilde{x}^{i+1} \in \tilde{\Omega}_n(t^{i+1})$ closest to x^{i+1}, by virtue of the induction assumption we have

$$\|x^{i+1} - \tilde{x}^{i+1}\| \le \tilde{\varepsilon}^{i+1}.$$

Hence we have the inequality (see Assertion 2.2)

$$\alpha(\tilde{F}^{-1}(t^i; t^{i+1}, x^{i+1}) - x^{i+1}, \tilde{F}^{-1}(t^i; t^{i+1}, \tilde{x}^{i+1}) - \tilde{x}^{i+1}) \le \Delta_n L \tilde{\varepsilon}^{i+1}. \quad (2.10)$$

We take the point

$$\bar{x}^i \in \tilde{F}^{-1}(t^i; t^{i+1}, x^{i+1})$$

as the closest to x^i; then the following inequality holds (see Assertion 2.1)

$$\|x^i - \bar{x}^i\| \le \omega(\Delta_n). \quad (2.11)$$

It follows from inequality (2.10) that there is a point

$$\tilde{x}^i \in \tilde{F}^{-1}(t^i; t^{i+1}, \tilde{x}^{i+1}),$$

which satisfies the inequality

$$\|(\bar{x}^i - x^{i+1}) - (\tilde{x}^i - \tilde{x}^{i+1})\| \le \Delta_n L \tilde{\varepsilon}^{i+1},$$

and so, also, the inequality

$$\|\bar{x}^i - \tilde{x}^i\| \le (1 + L\Delta_n)\tilde{\varepsilon}^{i+1}.$$

Then, bearing inequality (2.11) in mind, we have

$$\|x^i - \tilde{x}^i\| \le \tilde{\varepsilon}^i.$$

Since $x^i \in \Omega(t^i)$, the inclusion $x^i \in \Phi(t^i)$ holds; consequently, $\tilde{x}^t \in \Phi(t_i)_{\tilde{\varepsilon}^i}$ and so $\tilde{x}^i \in \tilde{\Omega}_n(t^i)$.

Since the choice of the point $x^i \in \Omega(t^i)$ is arbitrary, the inclusion $\Omega(t^i) \subset \tilde{\Omega}_n(t^i)_{\tilde{\varepsilon}^i}$ is proved.

Moreover, inclusion (2.9) is proved.

We will use inclusion (2.9) to prove the inclusion $\Omega \subset \tilde{\Omega}^0$.

When $t_* = \theta$ the inequalities $\Omega(t_*) = \Phi(\theta)$ and $\tilde{\Omega}^0(t_*) = \Phi(\theta)$ are satisfied. Consequently, the inclusion $\Omega(t_*) \subset \tilde{\Omega}^0(t_*)$ holds.

For any fixed $t_* < \theta$ we choose an arbitrary point $(t_*, x_*) \in \Omega$. The trajectory $x(t) \in X(t_*, x_*)$, which is viable in Φ, corresponds to it.

We fix the number n. Suppose $\eta_n = t_n(t_*)$. It follows from the inclusions $x(\eta_n) \in \Omega(\eta_n)$ and (2.9) that a point $x_n \in \tilde{\Omega}_n(\eta_n)$ exists, which satisfies the inequality

$$\|x(\eta_n) - x_n\| \le \tilde{\varepsilon}_n.$$

Then, using the inequality

$$\|(t_*, x_*) - (\eta_n, x_n)\| \le |t_* - \eta_n| + \|x_* - x(\eta_n)\| + \|x(\eta_n) - x_n\|$$

we have

$$\|(t_*, x_*) - (\eta_n, x_n)\| \le (1 + M)\Delta_n + \tilde{\varepsilon}_n$$

whence, by virtue of the limit relation $\lim \Delta_n = 0$ and Lemma 2.1, we obtain

$$(t_*, x_*) = \lim(\eta_n, x_n).$$

Consequently, $(t_*, x_*) \in \tilde{\Omega}^0$.

When $t_* < \theta$ the inclusion $\Omega(t_*) \subset \tilde{\Omega}^0(t_*)$ is proved.

The inclusion $\Omega \subset \tilde{\Omega}^0$ follows from relations $\Omega(\theta) \subset \tilde{\Omega}^0(\theta)$ and $\Omega(t_*) \subset \tilde{\Omega}^0(t_*)(t_* < \theta)$.

The equality $\Omega \subset \tilde{\Omega}^0$ follows from the inclusions $\tilde{\Omega}^0 \subset \Omega$, $\Omega \subset \tilde{\Omega}^0$. □

3. Discretization of the phase space

We will replace the phase space R^m by a certain γ-grid. A number of constructions will correspond to each subdivision Γ_n.

A. We will subdivide the space R^m into m-dimensional cubes B_j with centres b_j and vertices which are distant from the centres by an amount γ_n. We will choose the quantity γ_n so that it satisfies the inequality

$$\gamma_n \le \Delta_n^2.$$

The infinite set of centres b_j will be called the γ_n-grid of the space R^m, and we will denote it by $N^{\gamma_n}(R^m)$.

Suppose X_* is a certain compactum in R^m. We will isolate all the cubes B_j ($j = 1, 2, \ldots, J_0$), for which $B_j \cap X_* \neq \varnothing$, since the set X_* is bounded, and the number J_0 is finite. We will consider the centres b_j ($j = 1, 2, \ldots, J_0$) of these cubes. When $\varepsilon > 0$ we will assume

$$N^{\gamma_n}(X_*) = \{b_j : j = 1, 2, \ldots, J_0\}, \quad N_\varepsilon^{\gamma_n}(X_*) = N^{\gamma_n}(N^{\gamma_n}(X_*)_\varepsilon).$$

Note that

$$\alpha(X_*, N^{\gamma_n}(X_*)) \le \gamma_n.$$

B. We will assign a finite δ_n-grid to each set $F(t^*; t_*, x_*)$ ($(t_*, x_*) \in I \times R^m$, $t^* \in [t_*, \theta]$) using a certain rule

$$F_{\delta_n}(t^*; t_*, x_*) = \{f_k \in F(t^*; t_*, x_*) : k = 1, 2, \ldots, K_0\}$$

such that

$$\alpha(F(t^*; t_*, x_*), F_{\delta_n}(t^*; t_*, x_*)) \le \delta_n.$$

The number δ_n will be chosen to have any value which satisfies the inequality

$$\delta_n \le \Delta_n^2.$$

We will assume that

$$\tilde{F}_{\delta_n}^{-1}(t_*; t^*, x^*) = x^* - F_{\delta_n}(t^*; t_*, x^*)$$

$$\tilde{F}_{\delta_n}^{-1}(t_*; t^*, X^*) = \bigcup\{\tilde{F}_{\delta_n}^{-1}(t_*; t^*, x^*) : x^* \in X^*\}, \quad X^* \subset R^m.$$

Note that we have the estimate

$$\alpha(\tilde{F}^{-1}(t_*; t^*, x^*), \tilde{F}_{\delta_n}^{-1}(t_*; t^*, x^*)) \le \delta_n.$$

C. We will specify the recurrent sequence $\{\bar{\varepsilon}^i\}$ of number $\bar{\varepsilon}^i$

$$\bar{\varepsilon}_n = \gamma_n; \quad \bar{\varepsilon}^i = 2\gamma_n + \omega(\Delta_n) + \delta_n + (1 + L\Delta_n)\bar{\varepsilon}^{i+1}, \quad i = n-1, n-2, \ldots, 0.$$

Suppose $\bar{\varepsilon}_n$ is the greatest of the numbers $\{\bar{\varepsilon}^i\}$.

Lemma 3.1. *The following limit relation holds*

$$\lim \bar{\varepsilon}_n = 0.$$

The proof of this lemma is similar to the proof the Lemma 2.1 *and will not be given here.*

We will make the sequence $\{\bar{\Omega}_n(t^i)\}$ *of the sets* $\bar{\Omega}_n(t^i) \in N^{\gamma_n}(R^m)(t^i \in \Gamma_n)$, *specified recurrently, beginning from a finite instant* $t^n = \theta$, *correspond to each subdivision of* Γ_n.

Definition 3.1. We will assume that

$$\bar{\Omega}_n(\theta) = N^{\gamma_n}_{\bar{\varepsilon}^n}(\Phi(\theta)); \quad \bar{\Omega}_n(t^i) = N^{\gamma_n}_{\bar{\varepsilon}^i}(\Phi(t^i)) \cap N^{\gamma_n}(\tilde{F}^{-1}_{\delta_n}(t^i; t^{i+1}, \bar{\Omega}_n(t^{i+1}))),$$
$$i = n-1, n-2, \dots, 0.$$

Hence, the sequence $\{\bar{\Omega}_n(t^i)\}$ is a backwardly specified sequence of sets $\bar{\Omega}_n(t^i)$ on the grid $N^{\gamma_n}(R^m)$. We will determine the limit of this sequence when the diameter of the subdivision Γ_n approaches zero.

Definition 3.2. We will assume that $\bar{\Omega}^0$ is a set of all points $(t_*, x_*) \in I \times R^m$ for which we have the sequence

$$\{(\eta_n, x_n) : \eta_n = t_n(t_*), x_n \in \bar{\Omega}_n(\eta_n)\}$$

such that $(t_*, x_*) = \lim(\eta_n, x_n)$.

It follows from Definition 3.2 that $\bar{\Omega}^0 \subset \Phi$.

The set $\bar{\Omega}^0$ is non-empty, since the equality $\bar{\Omega}_n(\theta) = N^{\gamma_n}_{\bar{\varepsilon}^n}(\Phi(\theta))$ holds, and for the section $\bar{\Omega}^0(\theta) = \{x \in R^m : (\theta, x) \in \bar{\Omega}^0\}$ it is easy to prove the relation $\bar{\Omega}^0(\theta) = \Phi(\theta)$.

Theorem 3.1. *The set* $\bar{\Omega}^0$ *is the viability kernel of the generalized dynamical system (1.1) in the set* Φ.

The proof is largely similar to the proof of Theorem 2.1 (the difference caused by the presence of constructions from subsections A and B are of a technical character) and will not be given here.

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